

H-PACKING OF SOME INTERCONNECTION NETWORKS

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ABSTRACT

H-packing of a graph G is a set $\{H_1, H_2, H_3, H_4, \dots\}$ of vertex-disjoint subgraphs of G where each subgraph is isomorphic to H . H-packing of G is maximum, if it covers the greatest possible number of vertices of G and is called a perfect H-packing or an H-factor of G , if it covers all the vertices of G . The H-packing number of G is the maximum cardinality of the H-packing of G . Albert William and A. Shanthakumari [1] obtained the H-Packing Number of Butterfly Network. We obtain H-Factor of Hypercube and Pyramid Network and their packing numbers.

KEYWORDS: Matching, Packing, H-packing, Packing Number

Received: Oct 01, 2016; **Accepted:** Oct 26, 2016; **Published:** Nov 01, 2016; **Paper Id.:** IJMCARDEC20164

1. INTRODUCTION

The problem of covering the vertices of a given undirected graph with a maximum number of disjoint copies of the complete graph on two vertices, K_2 , is called the maximum matching problem. The problem of covering a graph with copies of graphs other than K_2 is called the graph packing problem. In this paper we find the H-Factor of some interconnection networks

Given a graph G and a subgraph H of G , a H-packing of G is a collection of vertex-disjoint copies of H in G . In other words, the H-packing of G is a set $\{H_1, H_2, H_3, H_4, \dots\}$ of vertex-disjoint subgraphs of G where each subgraph is isomorphic to H . An H-packing covers a vertex v of G if one of the subgraphs of the packing contains v . An H-packing of G is maximum, if it covers the greatest possible number of vertices of G and is called a perfect H-packing or an H-factor of G , if it covers all the vertices of G . The H-packing number of G is the maximum cardinality of the H-packing of G .

Over the past four decades, many research works have been pursued in packing of graphs [2]. When the graph H is a connected graph with at least three vertices, D.G. Kirkpatrick and P. Hell proved that the H-packing problem (H-factor problem) is NP-complete [3]. Albert William and A. Shanthakumari [1] obtained the H-Packing Number of Butterfly Networks.

Apart from theoretical interest, the graph packing problem is of practical interest in the areas of scheduling [3], wiring-board design, code optimization, exam scheduling and in the study of degree constraint subgraphs [4] and wireless sensor tracking [5].

In this paper, we obtain H-Packing or H-Factor of Hypercube and Pyramid Networks and their packing number.

2. HYPERCUBE NETWORK

The hypercube [6], is one of the most popular, versatile and efficient topological structures of

interconnection networks. The hypercube has many excellent features, and, thus becomes the first choice for the topological structure of parallel processing and computing systems [7]. The machines based on the hypercube have been implemented commercially such as the Cosmic Cube from Caltech [8], the iPSC /2 from Intel [9] and Connection Machines [10]. Parallel algorithms based on the hypercube have been developed [7]. The hypercube have been much studied in graph theory [11].

The topological structure of a hypercube network is the n -dimensional cube, shortly n -cube, whose graph-theoretic model is an undirected graph and denoted by Q_n . The vertex set V of Q_n consists of all binary sequence of length n on the set $\{0,1\}$, i.e.,

$$V = \{x_1x_2 \dots x_n : x_i \in \{0,1\}, i = 1,2, \dots, n\}.$$

Two vertices $x = x_1x_2 \dots x_n$ and $y = y_1y_2 \dots y_n$ are linked by an edge if and only if x and y differ exactly in one coordinate i.e. $\sum_{i=1}^n |x_i - y_i| = 1$.

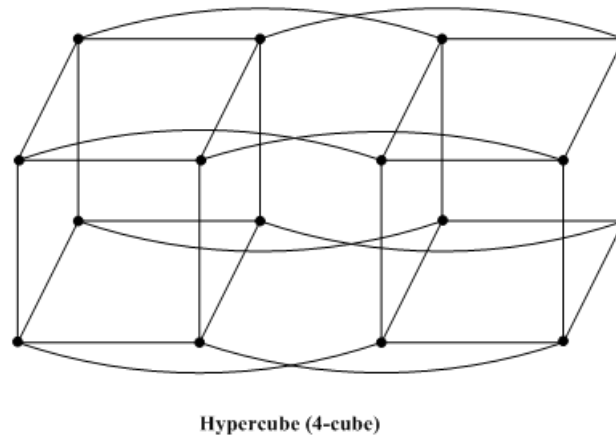


Figure 3.1: Hypercube Q_4 with Two Copies of Q_3

Theorem 2.1: Let G be a hypercube of order n . Let H be a subgraph of Q_n . If H is isomorphic to C_4 then there exist a H -factor of Q_n with $M_n(G, H) = 2^{n-2}$.

Proof:

We prove this theorem by induction on the dimension of the hypercube Q_n .

Base Case: $n = 2$. Q_2 is isomorphic to C_4 . and hence $M_2(G, H) = 1 = 2^{n-2}$.

Here the packing of Q_2 is perfect.

Assume that the theorem is true for Q_{n-1} .

Consider Q_n . Q_n contains exactly two copies of Q_{n-1} .

$$\therefore M_n(G, H) = 2 M_{n-1}(G, H)$$

$$= 2 \cdot \frac{2^{n-1}}{4} = 2^{n-2}.$$

Since $V(Q_n) = 2^n$, the packing is perfect.

Thus Q_n has a H- factor where $H \cong C_4$ and the Packing Number is $M_n(G, H) = 2^{n-2}$.

Corollary: Q_n has a H-factor K_2 where $M_n(G, H) = \frac{2^n}{2} = 2^{n-1}$ which is the matching number of Q_n .

Theorem 2.2: Let G be a hypercube of order n . Let H be subgraph of Q_n . If $H \cong S_{1,3}$, then there exists a H- factor of Q_n with $M_n(G, H) = 2^{n-2}$.

Proof

We prove the theorem by induction on the dimension of the hypercube Q_n .

Base case: $n = 3$. $V(Q_3)$ is partitioned into two sets of $S_{1,3}$.

$$\therefore M_3(G, H) = 2 = \frac{2^3}{4} = \frac{2^n}{4}, n = 3$$

Assume that the result is true for Q_{n-1} .

Consider Q_n . Q_n has exactly two copies of Q_{n-1} .

$$\therefore M_n(G, H) = 2 \cdot M_{n-1}(G, H)$$

$$= 2 \cdot \frac{2^{n-1}}{4} = \frac{2^n}{4} = 2^{n-2}$$

Since $V(Q_n) = 2^n$, the packing is perfect.

Thus Q_n has a H-factor, where $H \cong S_{1,4}$ and the Packing Number is $M_n(G, H) = 2^{n-2}$.

3. PYRAMID NETWORK

The pyramid network, suggested by Dyer and Rosenfeld [12], is one of the important structures in parallel computing [7] and image processing, [13]).In image processing, the pyramid networks are used as both hardware architectures and software structures. In parallel and network computing, a lot of parallel algorithms can be efficiently realized on the pyramid networks. For example, some parallel algorithms are realized in supercomputers like Gray T3D and T3E. Other parallel algorithms are realized by involving by several workstations, each workstation acting as a vertex in the pyramid network.

The vertex set of an n-dimensional pyramid network $PN(n)$ is

$$V(PN(n)) = \{(x, y, i): 1 \leq x, y \leq 2^i, 0 \leq i \leq n\}$$

For each fixed i ($0 \leq i \leq n$), $V_i = \{(x, y, i): 1 \leq x, y \leq 2^i\}$ is called a set of the vertices on level i . The subgraph induced by V_i is a mesh network $G(2^i, 2^i)$. For each $(x, y, i) \in V_i$ is adjacent to four vertices of V_{i+1} :

$$(2x - 1, 2y, i + 1), (2x, 2y, i + 1), (2x - 1, 2y - 1, i + 1), (2x, 2y - 1, i + 1)$$

The vertex $(1,0,0)$ is called the root of $PN(n)$. The graph shown in figure 5.2 is $PN(3)$.

Since $|V_i| = 2^i \cdot 2^i = 4^i$, The number of vertices of $PN(n)$ is

$$v(PN(n)) = 4^0 + 4^1 + 4^2 + \dots + 4^i + \dots + 4^n = \frac{1}{3}(4^{n+1} - 1).$$

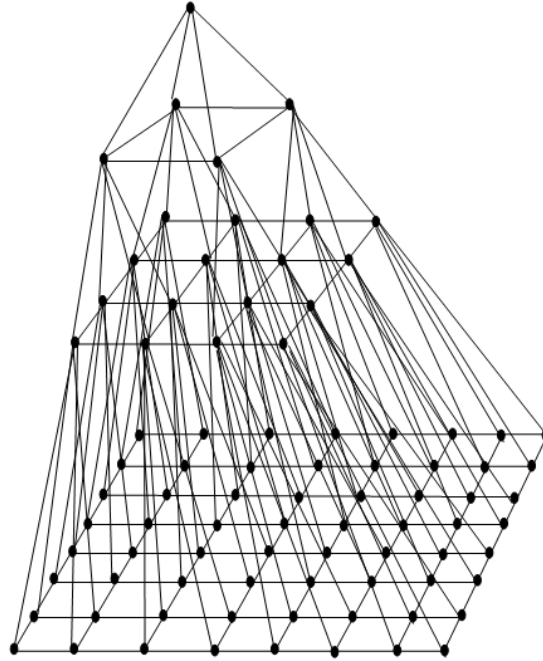


Figure 5.2: The Pyramid Network PN(3)

The number of edges on level i is $2^{i+1}(2^i - 1)$. Since the subgraph induced by V_i is a mesh network $G(2^i, 2^i)$. The number of edges between level i and $i - 1$ is equal to $|V_i|$.

Thus the number of the edges

$$\varepsilon(PN(n)) = \sum_{i=0}^n 2^{i+1}(2^i - 1) + \sum_{i=1}^n 4^i = 4^{n+1} - 2^{n+2} = 4(4^n - 2^n).$$

The minimum degree is 3, and the maximum degree is 9 for $n \geq 3$.

Theorem 3.1: Let G be an odd dimensional pyramid network $PN(n)$ and H be a subgraph of G . If H is isomorphic to $S_{1,4}$, then there exists a H -factor of $PN(n)$ with $M_n(G, H) = \frac{1}{3 \times 5} [4^{n+1} - 1]$.

Proof :

We prove this theorem by induction on the dimension of the pyramid network $PN(n)$, when n is odd.

Base case: $n = 1$. $PN(1)$ isomorphic to $S_{1,4}$.

$$M_1(G, H) = 1 = \frac{1}{3 \times 5} [4^2 - 1]$$

Hence the theorem is true for $n = 1$ (odd).

Assume that the theorem is true for $PN(n - 1)$, n is even (i.e., $n-1$ is odd)

To prove that the theorem is true for $PN(n + 1)$.

$$M_{n-1}(G, H) = \frac{1}{3 \times 5} [4^{(n-1)+1} - 1]$$

$$= \frac{1}{3 \times 5} [4^n - 1]$$

Since $PN(n+1)$ contains $PN(n-1)$ and 4^n sets of $S_{1,4}$.

$$\begin{aligned}
 M_{n+1}(G, H) &= M_{n-1}(G, H) + 4^n \\
 &= \frac{1}{3 \times 5} [4^{n-1+1} - 1] + 4^n \\
 &= 4^n \left[\frac{1}{15} + 1 \right] - \frac{1}{3 \times 5} \\
 &= \frac{4^n + 16}{15} - \frac{1}{15} = \frac{1}{3 \times 5} [4^{n+2} - 1]. \\
 \therefore V(PN(n)) &= \frac{1}{3} [4^{n+1} - 1], \text{ the packing is perfect and the packing number is} \\
 M_n(G, H) &= \frac{1}{3 \times 5} [4^{n+1} - 1].
 \end{aligned}$$

4. CONCLUSIONS

In this paper, we obtained the H-Factor of Hypercube and Pyramid Network and their packing number. Finding the H-Factor of Benes Network and its packing number is under investigation.

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